we first change the column $c,\ a,\ 1$ into

$$c,\quad 0,\quad \begin{vmatrix} 0 & b & 0 \\ b & 1 & a \\ c & a & 1 \end{vmatrix} \div (-b^2),$$

i. e., into

$$c,\quad 0,\quad 1-(ac\div b);$$

and so on, exactly as Professor Van Velzer does.

This example fortunately is easy, and the process as applied to it appears to the best advantage. It is desirable however to see the shady side as well, and for this purpose I give the curious identity

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ a_5+b_2 & a_1+b_3 & a_2+b_4 & a_3+b_5 & a_4+b_1 \\ a_4+b_3 & a_5+b_4 & a_1+b_5 & a_2+b_1 & a_3+b_2 \\ a_3+b_4 & a_4+b_5 & a_5+b_1 & a_1+b_2 & a_2+b_3 \\ a_2+b_5 & a_3+b_1 & a_4+b_2 & a_5+b_3 & a_1+b_4 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ a_5-b_2 & a_1-b_3 & a_2-b_4 & a_3-b_5 & a_4-b_1 \\ a_4-b_3 & a_5-b_4 & a_1-b_5 & a_2-b_1 & a_3-b_2 \\ a_3-b_4 & a_4-b_5 & a_5-b_1 & a_1-b_2 & a_2-b_3 \\ a_2-b_5 & a_3-b_1 & a_4-b_2 & a_5-b_3 & a_1-b_4 \end{vmatrix}$$

which possesses considerable interest in the theory of alternants.

Bishopton, Glasgow, Scotland, Oct. 1882.

INTEGRATION OF SOME GENERAL CLASSES OF TRIGONOMETRIC FUNCTIONS.

BY PROF. P. H. PHILBRICK, IOWA STATE UNIVERSITY, IOWA CITY.

[Continued from page 180, Vol. IX.]

$$\therefore \int\frac{dx}{(a+b\sec x)^n} = \int\frac{adx}{(a+b\sec x)^{n+1}} - \frac{\tan x\sec x}{(a+b\sec x)^{n+1}} + (n+1)b$$
$$\times\int\frac{\sec^2 x dx}{(a+b\sec x)^{n+2}} + 2\int\frac{\sec^3 x\, dx}{(a+b\sec x)^{n+1}} - (n+1)b\int\frac{\sec^4 xdx}{(a+b\sec x)^{n+2}}.$$

Now

$$\frac{\sec^2 x}{(a+b\sec x)^{n+2}} = \frac{1}{b^2}\left[\frac{1}{(a+b\sec x^n} - \frac{2a}{(a+b\sec x)^{n+1}} + \frac{a^2}{(a+b\sec x)^{n+2}}\right]$$

$$\frac{\sec^3 x}{(a+b\sec x)^{n+1}} = \frac{1}{b^3}\left[\frac{1}{(a+b\sec x)^{n-2}} - \frac{3a}{(a+b\sec x)^{n-1}} + \frac{3a^2}{(a+b\sec x)^n} - \frac{a^3}{(a+b\sec x)^{n+1}}\right]$$

$$\frac{\sec^4 x}{(a+b\sec x)^{n+2}} = \frac{1}{b^4}\left[\frac{1}{(a+b\sec x)^{n-2}} - \frac{4a}{(a+b\sec x)^{n-1}} + \frac{6a^2}{(a+b\sec x)^n} - \frac{4a^3}{(a+b\sec x)^{n+1}} + \frac{a4}{(a+b\sec x)^{n+2}}\right].$$

$$\therefore \int \frac{dx}{(a+b\sec x)^n} = \int \frac{a\,dx}{(a+b\sec x)^{n+1}} - \frac{\tan x \sec x}{(a+b\sec x)^{n+1}} + \frac{n+1}{b}\int \frac{dx}{(a+b\sec x)^n}$$

$$-\frac{2a(n+1)}{b}\int \frac{dx}{(a+b\sec x)^{n+1}} + \frac{a^2(n+1)}{b}\int \frac{dx}{(a+b\sec x)^{n+2}} + \frac{2}{b^3}\int \frac{dx}{(a+b\sec x)^{n-2}}$$

$$-\frac{6a}{b^3}\int \frac{dx}{(a+b\sec x)^{n-1}} + \frac{6a^2}{b^3}\int \frac{dx}{(a+b\sec x)^n} - \frac{2a^3}{b^3}\int \frac{dx}{(a+b\sec x)^{n+1}}$$

$$-\frac{n+1}{b^3}\int \frac{dx}{(a+b\sec x)^{n-2}} + \frac{4a(n+1)}{b^3}\int \frac{dx}{(a+b\sec x)^{n-1}} - \frac{6a^2(n+1)}{b^3}\int \frac{dx}{(a+b\sec x)^n}$$

$$+\frac{4a^3(n+1)}{b^3}\int \frac{dx}{(a+b\sec x)^{n+1}} - \frac{a^4(n+1)}{b^3}\int \frac{dx}{(a+b\sec x)^{n+2}}, \text{ or}$$

$$\int \frac{dx}{(a+b\sec x)^n} = -\frac{n-1}{b^3}\int \frac{dx}{(a+b\sec x)^{n-2}} + \frac{2a(2n-1)}{b^3}\int \frac{dx}{(a+b\sec x)^{n-1}}$$

$$+\frac{(n+1)(b^2-6a^2)+6a}{b^3}\int \frac{dx}{(a+b\sec x)^n} + \frac{ab^2(b-2n-2)+2a^3(2n+1)}{b^3}$$

$$\times \int \frac{dx}{(a+b\sec x)^{n+1}} - \frac{a^2(n+1)(a^2-b^2)}{b^3}\int \frac{dx}{(a+b\sec x)^{n+2}} - \frac{\tan x \sec x}{(a+b\sec x)^{n+1}}.$$

Transposing, dividing and writing $n-2$ for n we have:—

$$\int \frac{dx}{(a+b\sec x)^n} = -\frac{n-3}{(n-1)a^2(a^2-b^2)}\int \frac{dx}{(a+b\sec x)^{n-4}} + \frac{2(2n-5)}{a(n-1)(a^2-b^2)}$$

$$\times \int \frac{dx}{(a+b\sec x)^{n-3}} + \frac{(n-1)(b^2-6a^2)+6a-b^3}{a^2(n-1)(a^2-b^2)}\int \frac{dx}{(a+b\sec x)^{n-2}}$$

$$+\frac{b^2(b-2n+2)+2a^2(2n-3)}{a(n-1)(a^2-b^2)}\int \frac{dx}{(a+b\sec x)^{n-1}} - \frac{b^3}{a^2(n-1)(a^2-b^2)}\frac{\tan x \sec x}{(a+b\sec x)^{n-1}} \qquad (5)$$

In a similar manner we may find:—

$$\int \frac{dx}{(a+b\,\mathrm{co'c}\,x)^n} = -\frac{n-3}{(n-1)(a^2-b^2)}\int \frac{dx}{(a+b\,\mathrm{co'c}\,x)^{n-4}} + \frac{2(2n-5x)}{a(n-1)(a^2-b^2)}$$

$$\times \int \frac{dx}{(a+b\,\mathrm{cosec}\,x)^{n-3}} + \frac{(n-1)(b^2-6a^2)+6a-b^3}{a^2(n-1)(a^2-b^2)}\int \frac{dx}{(a+b\,\mathrm{cosec}\,x)^{n-2}}$$

$$+\frac{b^2(b-2n+2)+2a^2(2n-3)}{a(n-1)(a^2-b^2)}\int \frac{dx}{(a+b\,\mathrm{co'c}\,x)^{n-1}} + \frac{b^3}{a^2(n-1)(a^2-b^2)}\frac{\mathrm{cotan}\,x\,\mathrm{cosec}\,x}{(a+b\,\mathrm{co'c}\,x)^{n-1}} \qquad (6)$$

Now

$$\int \frac{\sin^m x\,dx}{(a+b\sin x)^n} = \frac{1}{b^m}\left[\int \frac{dx}{(a+b\tan x)^{n-m}} - ma\int \frac{dx}{(a+b\tan x)^{n-m+1}}\right.$$

$$\left.+\frac{m(m-1)}{1.2}a^2\int \frac{dx}{(a+b\tan x)^{n-m+2}} + \dots + a^m\int \frac{dx}{(a+b\tan x)^m}\right], \qquad (m)$$

the coefficients of the partial fractions within the brackets *being the successive terms* of $(1-a)^m$. Similarly for other Trigonometric functions. Hence

the preceding equations enable us to integrate the general form

$$\frac{\sin^m x dx}{(a+b\sin x)^n}$$

involving any of the trigonometric functions.

It is important to notice that m may have any value, positive or negative as I will presently show, and hence the equations apply also to

$$\int\frac{dx}{\sin^m x(a+b\sin x)^n},$$ etc., constituting a general class.

If in (m), $m > n$ some of the terms are of the form $\int\frac{dx}{(a+b\sin x)^{-p}} = \int(a+b\sin x)^p dx$, and they must be expanded and integrated, the others being integrated by the formulas above.

Let $y = 90° - x$, then $\cos y = \sin x$, $dy = -dx$.

$$\therefore \int\frac{dy}{(a+b\cos y)^n} = -\int\frac{dx}{(a+b\sin x)^n} = -\frac{b\cos x}{(n-1)(a^2-b^2)(a+b\sin x)^{n-1}}$$

$$-\frac{2n-3}{(n-1)(a^2-b^2)}\int\frac{dx}{(a+b\sin x)^{n-1}}+\frac{n-2}{(n-1)(a^2-b^2)}\int\frac{dx}{(a+b\sin x)^{n-2}}$$

$$=-\frac{b\sin y}{(n-1)(a^2-b^2)(a+b\cos y)^{n-1}}+\frac{2n-3}{(n-1)(a^2-b^2)}\int\frac{dy}{(a+b\cos y)^{n-1}}$$

$$-\frac{n-2}{(n-1)(a^2-b^2)}\int\frac{dy}{(a+b\cos y)^{n-2}}, \tag{7}$$

which is the same as eq. (2).

Hence the formula for $\int F(\cos x)dx$ may be obt'd from that for $\int F(\sin x)dx$ by replacing $\sin x$ with $\cos x$, etc., and changing the signs of the terms without the sign of integration. The formulas for $\int F(\cot x)dx$ and $\int F(\operatorname{cosec} x)dx$ may be writ'n in a similar manner from those of $\int F(\tan x)dx$ and $\int F(\sec x)dx$.

Again, $\operatorname{cosec} x = (1\div\sin x)$, $\sec x = (1\div\cos x)$ and $\cot x = (1\div\tan)$.

$$\therefore \frac{\operatorname{cosec}^m x\,dx}{(a+b\operatorname{cosec} x)^n} = \frac{\sin^{n-m} x\,dx}{(b+a\sin x)^n}. \tag{10}$$

$$\frac{\sec^m x\,dx}{(a+b\sec x)^n} = \frac{\cos^{n-m} x\,dx}{(b+a\cos x)^n}. \tag{11}$$

$$\frac{\cot^m x\,dx}{(a+b\cot x)^n} = \frac{\tan^{n-m} x\,dx}{(b+a\tan x)^n}. \tag{12}$$

These eq's enable us to integrate the three former forms in terms of the three latter, or *vice versa*, whether m is *positive* or *negative*. See eq. (m) and the remarks following.

Example (1). $$\int\frac{dx}{\sec x(a+b\sec x)} = \int\frac{\cos^2 x\,dx}{(b+a\cos x)}$$

$$=\frac{1}{a^2}\left[\int(b+a\cos x)dx - 2b\int dx + b^2\int\frac{dx}{(b+a\cos x)}\right] \tag{13}$$

$$= \frac{1}{a^2}\left[\int(b + a\cos x)dx - 2b\int dx + b^2\int\frac{\sec x\,dx}{(a+b\sec x)}\right] \tag{14}$$

Eqn. (13) may be used, giving results in terms of the cosine throughout, or (14) may be used. Using (13) we have:

$$\int\frac{dx}{\sec x(a+b\sin x)} = \frac{\sin x}{a} - \frac{bx}{a^2} + \frac{2b^2}{(b^2-a^2)^{1/2}}\tan^{-1}\left[\left(\frac{b-a}{b+a}\right)^{\frac{1}{2}}\tan\frac{x}{2}\right].$$

Example (2). $\int\frac{dx}{\tan x(a+b\tan x)} = \int\frac{\cot^2 x\,dx}{(b+a\cot x)}$

$$= \frac{1}{a^2}\left[\int(b+a\cot x)dx - 2b\int dx + b^2\int\frac{dx}{(b+a\cot x)}\right] \tag{16}$$

or
$$= \frac{1}{a^2}\left[\int(b+a\cot x)dx - 2b\int dx + b^2\int\frac{\tan x\,dx}{a+b\tan x)}\right] \tag{17}$$

which are in known terms and may be treated similarly to the above.

Example (3). $\int\frac{\cos x\,dx}{(a+b\cos x)^3} = \frac{1}{b}\left[\int\frac{dx}{(a+b\cos x)^2} + a\int\frac{dx}{a+b\cos x)^3}\right].$

Now $a\int\frac{dx}{(a+b\cos x)^3} = A - B\int\frac{dx}{(a+b\cos x)^2} + C\int\frac{dx}{(a+b\cos x)};$

$$= A - B\left(D - E\int\frac{dx}{(a+b\cos x)}\right) + C\int\frac{dx}{(a+b\cos x)}$$

$$= A - BD + (BE+C)\int\frac{dx}{(a+b\cos x)}.$$

$$\therefore \int\frac{dx}{(a+b\cos x)^2} - a\int\frac{dx}{(a+b\cos x)^3} = D - E\int\frac{dx}{(a+b\cos x)} - A + BD$$

$$-(BE+C)\int\frac{dx}{(a+b\cos x)} = D + BD - A - (C+E+BE)\int\frac{dx}{(a+b\cos x)}.$$

In the above $A = \frac{ab\sin x}{2(b^2-a^2)(a+b\cos x)^2}$, $B = \frac{3a^2}{2(b^2-a^2)}$, $C = \frac{a}{2(b^2-a^2)}$,

$D = \frac{b\sin x}{(b^2-a^2)(a+b\cos x)}$, $E = \frac{a}{b^2-a^2}$; $\therefore BD = \frac{3a^2b\sin x}{2(b^2-a^2)^2(a+b\cos x)}$

and $BE = \frac{3a^3}{2(b^2-a^2)^2}$. Therefore

$$\int\frac{\cos x\,dx}{(a+b\cos x)^3} = \frac{a\sin x(2a^2+b^2) + b\sin x\cos x(a^2+2b^2)}{2(b^2-a^2)(a+b\cos x)^2}$$

$$-\frac{3ab}{2(b^2-a^2)}\int\frac{dx}{(a+b\cos x)}$$

$$= \frac{a\sin x(2a^2+b^2) + b\sin x\cos x(a^2+2b^2)}{2(b^2-a^2)(a+b\cos x)^2} + \frac{3ab}{(a^2-b^2)^{1/2}}\tan^{-1}\left[\left(\frac{a-b}{a+b}\right)^{\frac{1}{2}}\tan\frac{x}{2}\right]$$

where $a > b$.

Other Forms.

$$\int\frac{\sin^m x\cos^r x\,dx}{(a+b\cos x)^n}=\int\frac{\sin^m x\cos^{r-n}x\,dx}{(b+a\sec x)^n}=\int\frac{(1-\cos^2 x)^{\frac{1}{2}m}\cos^r x\,dx}{(a+b\cos x)^n} \qquad (a)$$

$$=-\int\frac{(1-\cos^2 x)^{\frac{1}{2}(m-1)}\cos^r x\,d\cos x}{(a+b\cos x)^n}. \qquad (b)$$

If m is even the right-hand member of (a) may be developed into a finite number of terms and integrated by the preceding formulas. If m is odd eq. (b) may be developed and will take the form of a rational fraction, with or without a series of mononomial terms, and may be integrated accordingly; or since each term would be of the form: $z^m(a+bz)^{-n}dz$, the exponent within the parentheses being *unity*, the successive terms may be integrated by substitution.

$$\frac{\sin^m x\cos^r x\,dx}{(a+b\sin x)^n}=\frac{\sin^{m-n}x\cos^r x\,dx}{(b+a\operatorname{cosec} x)^n}=\frac{(1-\sin^2 x)^{\frac{1}{2}r}\sin^m x\,dx}{(a+b\sin x)^n}. \qquad (c)$$

$$=\frac{(1-\sin^2 x)^{\frac{1}{2}(r-1)}\sin^m x\,d\sin x}{(a+b\sin x)^n}. \qquad (d)$$

These may be treated the same as (a) and (b). If $n=0$ in (a) and (b) we have the more special form:

$$\sin^m x\cos^r x\,dx=(1-\cos^2 x)^{\frac{1}{2}m}\cos^r x\,dx,\text{ if } m \text{ is even, or}$$
$$=(1-\cos^2 x)^{\frac{1}{2}(m-1)}\cos^r x\,d\cos x,\text{ if } m \text{ is odd.}$$

This may be developed and at once integrated which process is preferable to that involving the usual formu's for this case, formulas which provide for special sub-cases and involve successive integrals of different orders.

The above includes also the more special case in which $m=0$ or $r=0$.

$$\int\frac{\cos^m x\,dx}{(a+b\tan x)^n}=\int\frac{\cos^{m-1}x\cos x\,dx}{(a+b\tan x)^n}.\quad\text{Let }\cos x\,dx=dv,\ \sin x=v,$$

$$\frac{\cos^{m-1}x}{(a+b\tan x)^n}=u,\ -(m-1)\frac{\cos^{m-2}x\sin x\,dx}{(a+b\tan x)^n}-nb\frac{\cos^{m-1}x\sec^2 x\,dx}{(a+b\tan x)^{n+1}}=du.$$

$$\therefore vdu=-(m-1)\frac{\cos^{m-2}x\sin x\,dx}{(a+b\tan x)^n}-nb\frac{\cos^m x\tan x(1+\tan^2 x)dx}{(a+b\tan x)^{n+1}}$$

$$=-nb\frac{\cos^m x\tan x}{(a+b\tan x)^{n+1}}-a(m-1)\frac{\cos^m x\tan^2 x}{(a+b\tan x)^{n+1}}$$
$$-b(m+n-1)\frac{\cos^m x\tan^3 x}{(a+b\tan x)^{n+1}}$$

$$=-\frac{n\cos^m x}{(a+b\tan x)^n}+\frac{an\cos^m x}{(a+b\tan x)^{n+1}}-\frac{a(m-1)}{b^2}\frac{\cos^m x}{(a+b\tan x)^{n-1}}$$
$$+\frac{2a^2(m-1)}{b^2}\frac{\cos^m x}{(a+b\tan x)^n}-\frac{a^3(m-1)}{b^2}\frac{\cos^m x}{(a+b\tan x)^{n+1}}$$

$$-\frac{m+n-1}{b^2}\frac{\cos^m x}{(a+b\tan x)^{n-2}}+\frac{3a(m+n-1)}{b^2}\frac{\cos^m x}{(a+b\tan x)^{n-1}}$$

$$-\frac{3a^2(m+n-1)}{b^2}\frac{\cos^m x}{(a+b\tan x)^n}+\frac{a^3(m+n-1)}{b^2}\frac{\cos^m x}{(a+b\tan x)^{n+1}};$$

$$=-\frac{m+n-1}{b^2}\frac{\cos^m x}{(a+b\tan x)^{n-2}}+\frac{a(2m+3n-2)}{b^2}\frac{\cos^m x}{(a+b\tan x)^{n-1}}$$

$$-\frac{a^2(m+3n-1)+b^2n}{b^2}\frac{\cos^m x}{(a+b\tan x)^n}+\frac{an(a^2+b^2)}{b^2}\frac{\cos^m x}{(a+b\tan x)^{n+1}}.$$

$$\therefore\int\frac{\cos^m x\,dx}{(a+b\tan x)^n}=\frac{\cos^{m-1}x\sin x}{(a+b\tan x)^n}+\frac{m+n-1}{b^2}\int\frac{\cos^m x\,dx}{(a+b\tan x)^{n+2}}$$

$$-\frac{a(2m+3n-2)}{b^2}\int\frac{\cos^m x\,dx}{(a+b\tan x)^{n-1}}+\frac{a^2(m+3n-1)+b^2n}{b^2}\int\frac{\cos^m x\,dx}{(a+b\tan x)^n}$$

$$-\frac{an(a^2+b^2)}{b^2}\int\frac{\cos^m x\,dx}{(a+b\tan x)^{n+1}}.$$

Transposing, dividing by $\frac{an(a^2+b^2)}{b^2}$, and writing $n-1$ for n we finally get:—

$$\int\frac{\cos^m x\,dx}{(a+b\tan x)^n}=\frac{b^2}{a(n-1)\,(a^2-b^2)}\frac{\cos^{m-1}x\sin x}{(a+b\tan x)^{n-1}}$$

$$+\frac{a^2(m+3n-4)+b^2(n-2)}{a(n-1)(a^2+b^2)}\int\frac{\cos^m x\,dx}{(a+b\tan x)^{n-1}}$$

$$-\frac{(2m+3n-5)}{(n-1)(a^2+b^2)}\int\frac{\cos^m x\,dx}{(a+b\tan x)^{n-2}}+\frac{m+n-2}{a(n-1)\,(a^2+b^2)}\int\frac{\cos^m x\,dx}{(a+b\tan x)^{n-3}}. \quad (e)$$

Now

$$\int\frac{\sin^m x\,dx}{(a+b\tan x)^n}=\int\frac{\cos^m x\tan^m x\,dx}{(a+b\tan x)^n}.$$

$\frac{\tan^m x}{(a+b\tan x)^n}$ may be separated into a series of partial fractions, each may be multiplied by $\cos^m x\,dx$ and integrated by eq. (e).

The above includes

$$\frac{\sin^m x\cos^r x}{(a+b\tan x)^n}=\frac{\cos^{m+r}x\tan^m x}{(a+b\tan x)^n}.$$

Again

$$\int\frac{\sin^m x\cos^r x\,dx}{(a+b\cot x)^n}=\int\frac{\cos^{m+r-n}x\tan^{m+n}x\,dx}{(b+a\tan x)^n},$$

which may be integrated by the above, and which includes special cases in which $r=0$ or $m=0$.